PERIODIC HOMOGENIZATION OF SCHRÖDINGER TYPE EQUATIONS WITH RAPIDLY OSCILLATING POTENTIAL

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Abstract. This paper is devoted to the homogenization of Shrödinger type equations with periodically oscillating coefficients of the diffusion term, and a rapidly oscillating periodic time-dependent potential. One convergence theorem is proved and we derive the macroscopic homogenized model. Our approach is the well known two-scale convergence method.

1. Introduction

Let us consider a smooth bounded open subset Ω of $\mathbb{R}^N_x$ (the N-numerical space $\mathbb{R}^N$ of variables $x=(x_1,...,x_N)$), where N is a given positive integer, and let T and ε be real numbers with $T>0$ and $0<\varepsilon<1$. We consider the partial differential operator

$$\mathcal{A}^\varepsilon=-\sum_{i,j=1}^{N}\frac{\partial}{\partial x_i}\left(a^\varepsilon_{ij}\frac{\partial}{\partial x_j}\right)$$

in Ω, where $a^\varepsilon_{ij}(x)=a_{ij}\left(\frac{x}{\varepsilon}\right)\quad(x\in\Omega)$, $a_{ij}\in L^\infty\left(\mathbb{R}^N_y;\mathbb{R}\right)$ $(1\leq i,j\leq N)$ with

$$a_{ij}=a_{ji}, \tag{1.1}$$

and the assumption that there exists a constant $\alpha>0$ such that

$$\operatorname{Re}\sum_{i,j=1}^{N}a_{ij}(y,\tau)\zeta_j\overline{\zeta_i}\geq\alpha\left|\zeta\right|^2\text{ for all }\zeta=\left(\zeta_j\right)\in\mathbb{C}^N\text{ and} \tag{1.2}$$

for almost all $y\in\mathbb{R}^N$, where $\mathbb{R}^N_y$ is the N-numerical space $\mathbb{R}^N$ of variables $y=(y_1,...,y_N)$, and where $\left|\cdot\right|$ denotes the Euclidean norm in $\mathbb{C}^N$. Let us consider for fixed $0<\varepsilon<1$, the following initial boundary value problem:

$$\mathbf{i}\frac{\partial u_\varepsilon}{\partial t}+\mathcal{A}^\varepsilon u_\varepsilon+\frac{1}{\varepsilon}\mathcal{V}^\varepsilon u_\varepsilon=f\text{ in }\Omega\times]0,T[\tag{1.3}$$

$$u_\varepsilon=0\text{ on }\partial\Omega\times]0,T[\tag{1.4}$$

$$u_\varepsilon(0)=u^0\text{ in }\Omega, \tag{1.5}$$

where $\mathcal{V}^\varepsilon(x,t)=\mathcal{V}\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon}\right)$ is a real potential with $\mathcal{V}\in L^\infty\left(\mathbb{R}^N_y\times\mathbb{R}_\tau;\mathbb{R}\right)$, $\mathbb{R}_\tau$ being the numerical space $\mathbb{R}$ of variables τ, and where $f\in L^2\left(0,T;L^2(\Omega)\right)$, $u^0\in L^2(\Omega)$. In view of (1.1)-(1.2), we will show later that the initial boundary value problem (1.3)-(1.5) admits a unique solution in $\mathcal{C}\left([0,T];H^1_0(\Omega)\right)\cap\mathcal{C}^1\left([0,T];L^2(\Omega)\right)$, provided some regularity assumptions on f and u^0, and some hypothesis on $\mathcal{V}$. The

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aim here is to investigate the limiting behaviour of u_ε solution of (1.3)-(1.5) when ε goes to zero, under the periodicity hypotheses on the coefficients a_{ij} and the potential $\mathcal{V}$, and the assumption that the mean value of $\mathcal{V}$ is null.

The asymptotic analysis of boundary value problems with rapidly oscillating potential has been studied for the first time in the book of Bensoussan, Lions and Papanicolaou [2] using the asymptotic expansions. Recently, Allaire and Piatnitski in [1] have investigated the homogenization of a Schrödinger equation with a large periodic potential scaled as ε^{-2}, using the two-scale convergence method combined with the bloch waves decomposition.

This paper deals with the homogenization of an evolution problem with time-dependent coefficients and large rapidly oscillating potential via the two-scale convergence. Clearly, in our study we present an other point of view concerning the asymptotic analysis of the Schrödinger model, when the potential is scaled as ε^{-1}. In this paper, the derived macroscopic homogenized model is given by (3.31)-(3.33), while the equations at the microscopic scale are given by (3.27)-(3.28) and the global equation (including the macroscopic and the microscopic scales) by (3.15).

This study is motivated by the fact that the asymptotic analysis of (1.3)-(1.5) is connected with the modelling of the wave function for a particle submitted to a potential. Let us note that the classical Schrödinger equation corresponds to the choice $\mathcal{A}^\varepsilon = -\Delta$.

Unless otherwise specified, vector spaces throughout are considered over the complex field, $\mathbb{C}$, and scalar functions are assumed to take complex values. Let us recall some basic notation. If X and F denote a locally compact space and a Banach space, respectively, then we write $\mathcal{C}(X;F)$ for continuous mappings of X into F, and $\mathcal{B}(X;F)$ for those mappings in $\mathcal{C}(X;F)$ that are bounded. We shall assume $\mathcal{B}(X;F)$ to be equipped with the supremum norm $\|u\|_\infty = \sup_{x\in X}\|u(x)\|$ ($\|\cdot\|$ denotes the norm in F). For shortness we will write $\mathcal{C}(X) = \mathcal{C}(X;\mathbb{C})$ and $\mathcal{B}(X) = \mathcal{B}(X;\mathbb{C})$. Likewise in the case when $F = \mathbb{C}$, the usual spaces $L^p(X;F)$ and $L^p_{loc}(X;F)$ (X provided with a positive Radon measure) will be denoted by $L^p(X)$ and $L^p_{loc}(X)$, respectively. Finally, the numerical space $\mathbb{R}^N$ and its open sets are each provided with Lebesgue measure denoted by $dx = dx_1...dx_N$.

The rest of the paper is organized as follows. Section 2 is devoted to some preliminary results on the two-scale convergence, whereas in Section 3 one convergence theorem is established for (1.3)-(1.5).

2. Preliminaries

We set $Y = \left(-\frac{1}{2},\frac{1}{2}\right)^N$, Y considered as a subset of $\mathbb{R}^N_y$ (the space $\mathbb{R}^N$ of variables $y = (y_1,...,y_N)$). We set also $Z = \left(-\frac{1}{2},\frac{1}{2}\right)$, Z considered as a subset of $\mathbb{R}_\tau$ (the space $\mathbb{R}$ of variables τ).

Let us first recall that a function $u \in L^1_{loc}\left(\mathbb{R}^N_y \times \mathbb{R}_\tau\right)$ is said to be $Y\times Z$-periodic if for each $(k,l) \in \mathbb{Z}^N \times \mathbb{Z}$ ($\mathbb{Z}$ denotes the integers), we have $u(y+k,\tau+l) = u(y,\tau)$ almost everywhere (a.e.) in $(y,\tau) \in \mathbb{R}^N \times \mathbb{R}$. If in addition u is continuous, then the preceding equality holds for every $(y,\tau) \in \mathbb{R}^N \times \mathbb{R}$, of course. The space of all $Y\times Z$-periodic continuous complex functions on $\mathbb{R}^N_y \times \mathbb{R}_\tau$ is denoted by $\mathcal{C}_{per}(Y\times Z)$; that of all $Y\times Z$-periodic functions in $L^p_{loc}\left(\mathbb{R}^N_y \times \mathbb{R}_\tau\right)$ ($1 \le p \le \infty$) is denoted by $L^p_{per}(Y\times Z)$. $\mathcal{C}_{per}(Y\times Z)$ is a Banach space under the supremum

norm on $\mathbb{R}^N \times \mathbb{R}$, whereas $L^p_{per}(Y \times Z)$ is a Banach space under the norm

$$\|u\|_{L^p(Y\times Z)} = \left(\int_Z \int_Y |u(y,\tau)|^p \, dyd\tau\right)^{\frac{1}{p}} \quad \left(u \in L^p_{per}(Y\times Z)\right).$$

The space $H^1_\#(Y)$ of Y-periodic functions $u \in H^1_{loc}\left(\mathbb{R}^N_y\right) = W^{1,2}_{loc}\left(\mathbb{R}^N_y\right)$ such that $\int_Y u(y)\,dy = 0$ will be of our interest in this study. Provided with the gradient norm,

$$\|u\|_{H^1_\#(Y)} = \left(\int_Y |\nabla_y u|^2 \, dy\right)^{\frac{1}{2}} \quad \left(u \in H^1_\#(Y)\right),$$

where $\nabla_y u = \left(\frac{\partial u}{\partial y_1}, ..., \frac{\partial u}{\partial y_N}\right)$, $H^1_\#(Y)$ is a Hilbert space. We will also need the space $L^2_{per}\left(Z; H^1_\#(Y)\right)$ with the norm

$$\|u\|_{L^2_{per}\left(Z;H^1_\#(Y)\right)} = \left(\int_Z \int_Y |\nabla_y u(y,\tau)|^2 \, dyd\tau\right)^{\frac{1}{2}} \quad \left(u \in L^2_{per}\left(Z; H^1_\#(Y)\right)\right)$$

which is a Hilbert space.

Before we can recall the concept of two-scale convergence, let us introduce one further notation. The letter E throughout will denote a family of real numbers $0 < \varepsilon < 1$ admitting 0 as an accumulation point. For example, E may be the whole interval $(0,1)$; E may also be an ordinary sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $0 < \varepsilon_n < 1$ and $\varepsilon_n \to 0$ as $n \to \infty$. In the latter case E will be referred to as a *fundamental sequence*.

Let Ω be a bounded open set in $\mathbb{R}^N_x$ and $Q = \Omega \times]0,T[$ with $T \in \mathbb{R}^*_+$, and let $1 \le p < \infty$.

Definition 2.1. *A sequence* $(u_\varepsilon)_{\varepsilon\in E} \subset L^p(Q)$ *is said to:*

(i) weakly two-scale converge in $L^p(Q)$ *to some* $u_0 \in L^p\left(Q; L^p_{per}(Y\times Z)\right)$ *if as* $E \ni \varepsilon \to 0$*,*

$$\int_Q u_\varepsilon(x,t)\,\psi^\varepsilon(x,t)\,dxdt \to \int\int\int_{Q\times Y\times Z} u_0(x,t,y,\tau)\,\psi(x,t,y,\tau)\,dxdtdyd\tau \tag{2.1}$$

for all $\psi \in L^{p'}(Q; \mathcal{C}_{per}(Y\times Z))$ $\left(\frac{1}{p'} = 1 - \frac{1}{p}\right)$, *where* $\psi^\varepsilon(x,t) = \psi\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right)$ $((x,t)\in Q)$*;*

(ii) strongly two-scale converge in $L^p(Q)$ *to some* $u_0 \in L^p\left(Q; L^p_{per}(Y\times Z)\right)$ *if the following property is verified:*

$$\left\{\begin{array}{c} \text{Given } \eta > 0 \text{ and } v \in L^p(Q; \mathcal{C}_{per}(Y\times Z)) \text{ with} \\ \|u_0 - v\|_{L^p(Q\times Y\times Z)} \le \frac{\eta}{2}, \text{ there is some } \alpha > 0 \text{ such} \\ \text{that } \|u_\varepsilon - v^\varepsilon\|_{L^p(Q)} \le \eta \text{ provided } E \ni \varepsilon \le \alpha. \end{array}\right.$$

We will briefly express weak and strong two-scale convergence by writing $u_\varepsilon \to u_0$ in $L^p(Q)$-weak 2-s and $u_\varepsilon \to u_0$ in $L^p(Q)$-strong 2-s, respectively.

Remark 2.1. *It is of interest to know that if* $u_\varepsilon \to u_0$ *in* $L^p(Q)$*-weak 2-*s*, then (2.1) holds for* $\psi \in \mathcal{C}\left(\overline{Q}; L^\infty_{per}(Y\times Z)\right)$. *See* [8, Proposition 10] *for the proof.*

For more details about the two-scale convergence the reader can refer to [5]. However, we recall below two fundamental results. First of all, let

$$\mathcal{Y}(0,T) = \left\{v \in L^2\left(0,T; H^1_0(\Omega)\right) : v' \in L^2\left(0,T; H^{-1}(\Omega)\right)\right\}.$$

$\mathcal{Y}(0,T)$ is provided with the norm

$$\|v\|_{\mathcal{Y}(0,T)} = \left(\|v\|^2_{L^2\left(0,T;H^1_0(\Omega)\right)} + \|v'\|^2_{L^2(0,T;H^{-1}(\Omega))}\right)^{\frac{1}{2}} \qquad (v \in \mathcal{Y}(0,T))$$

which makes it a Hilbert space.

Theorem 2.1. *Assume that $1 < p < \infty$ and further E is a fundamental sequence. Let a sequence $(u_\varepsilon)_{\varepsilon\in E}$ be bounded in $L^p(Q)$. Then, a subsequence E' can be extracted from E such that $(u_\varepsilon)_{\varepsilon\in E'}$ weakly two-scale converges in $L^p(Q)$.*

Theorem 2.2. *Let E be a fundamental sequence. Suppose a sequence $(u_\varepsilon)_{\varepsilon\in E}$ is bounded in $\mathcal{Y}(0,T)$. Then, a subsequence E' can be extracted from E such that, as $E' \ni \varepsilon \to 0$,*

$$u_\varepsilon \to u_0 \text{ in } \mathcal{Y}(0,T)\text{-weak},$$

$$u_\varepsilon \to u_0 \text{ in } L^2(Q)\text{-weak 2-}s,$$

$$\frac{\partial u_\varepsilon}{\partial x_j} \to \frac{\partial u_0}{\partial x_j} + \frac{\partial u_1}{\partial y_j} \text{ in } L^2(Q)\text{-weak 2-}s \;\; (1 \le j \le N),$$

where $u_0 \in \mathcal{Y}(0,T)$, $u_1 \in L^2\left(Q; L^2_{per}\left(Z; H^1_\#(Y)\right)\right)$.

The proof of Theorem 2.1 can be found in, e.g., [5], [6], whereas Theorem 2.2 has its proof in, e.g., [8].

Let us prove the following lemma.

Lemma 2.1. *Let $(u_\varepsilon)_{\varepsilon\in E}$ be a bounded sequence in $\mathcal{Y}(0,T)$, where E is a fundamental sequence. There exists a subsequence E' extracted from E such that*

$$\int_Q \frac{1}{\varepsilon} u_\varepsilon \psi^\varepsilon dxdt \to \int_Q \int\int_{Y\times Z} u_1(x,t,y,\tau)\,\psi(x,t,y,\tau)\,dxdtdyd\tau \tag{2.2}$$

for all $\psi \in \mathcal{D}(Q)\otimes(\mathcal{C}_{per}(Y)/\mathbb{C})\otimes\mathcal{C}_{per}(Z)$ as $E' \ni \varepsilon \to 0$, where $u_1 \in L^2\left(Q; L^2_{per}\left(Z; H^1_\#(Y)\right)\right)$.

Proof. As $(u_\varepsilon)_{\varepsilon\in E}$ is a bounded sequence in $\mathcal{Y}(0,T)$, thanks to Theorem 2.2, there exists a subsequence E' extracted from E and functions $u_0 \in \mathcal{Y}(0,T)$, $u_1 \in L^2\left(Q; L^2_{per}\left(Z; H^1_\#(Y)\right)\right)$ such that

$$u_\varepsilon \to u_0 \text{ in } \mathcal{Y}(0,T)\text{-weak},$$

$$u_\varepsilon \to u_0 \text{ in } L^2(Q)\text{-weak 2-}s, \tag{2.3}$$

$$\frac{\partial u_\varepsilon}{\partial x_j} \to \frac{\partial u_0}{\partial x_j} + \frac{\partial u_1}{\partial y_j} \text{ in } L^2(Q)\text{-weak 2-}s \;\; (1 \le j \le N), \tag{2.4}$$

as $E' \ni \varepsilon \to 0$. Let $\theta \in \mathcal{D}(Q)\otimes\mathcal{C}^\infty_{per}(Y)\otimes\mathcal{C}_{per}(Z)$. We have

$$\frac{1}{\varepsilon}(\Delta_y\theta)^\varepsilon = \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\frac{\partial\theta}{\partial y_i}\right)^\varepsilon - \sum_{i=1}^N \left(\frac{\partial^2\theta}{\partial x_i \partial y_i}\right)^\varepsilon,$$

as is easily seen by observing that

$$\frac{\partial\Phi^\varepsilon}{\partial x_i} = \left(\frac{\partial\Phi}{\partial x_i}\right)^\varepsilon + \frac{1}{\varepsilon}\left(\frac{\partial\Phi}{\partial y_i}\right)^\varepsilon, \qquad \Phi \in \mathcal{C}^1\left(Q\times\mathbb{R}^N_y\times\mathbb{R}_\tau\right).$$

Hence,

$$\int_Q \frac{1}{\varepsilon} u_\varepsilon \left(\Delta_y \theta\right)^\varepsilon dxdt = -\int_Q \nabla_x u_\varepsilon \cdot \left(\nabla_y \theta\right)^\varepsilon dxdt - \int_Q u_\varepsilon \sum_{i=1}^N \left(\frac{\partial^2 \theta}{\partial x_i \partial y_i}\right)^\varepsilon dxdt, \tag{2.5}$$

where the dot denotes the Euclidean inner product. On the other hand, according to (2.3) and (2.4) we have

$$\int_Q u_\varepsilon \left(\frac{\partial^2 \theta}{\partial x_i \partial y_i}\right)^\varepsilon dxdt \to \int_Q u_0 \left(\int\int_{Y\times Z} \frac{\partial^2 \theta}{\partial x_i \partial y_i} dyd\tau\right) dxdt = 0$$

and

$$\int_Q \nabla_x u_\varepsilon \cdot \left(\nabla_y \theta\right)^\varepsilon dxdt \to \int\int\int_{Q\times Y\times Z} \left(\nabla_x u_0 + \nabla_y u_1\right) \cdot \nabla_y \theta dxdtdyd\tau$$

as $E' \ni \varepsilon \to 0$. Therefore, on letting $E' \ni \varepsilon \to 0$ in (2.5), one has

$$\int_Q \frac{1}{\varepsilon} u_\varepsilon \left(\Delta_y \theta\right)^\varepsilon dxdt \to \int\int\int_{Q\times Y\times Z} u_1 \Delta_y \theta dxdtdyd\tau.$$

With this in mind, let $\psi \in \mathcal{D}(Q) \otimes \left(\mathcal{C}^\infty_{per}(Y)/\mathbb{C}\right) \otimes \mathcal{C}_{per}(Z)$, i.e.,

$$\psi = \sum_{i\in I} \varphi_i \otimes \psi_i \otimes \chi_i$$

with $\varphi_i \in \mathcal{D}(Q)$, $\psi_i \in \mathcal{C}^\infty_{per}(Y)/\mathbb{C}$ and $\chi_i \in \mathcal{C}_{per}(Z)$, where I is a finite set (depending on ψ). For any $i \in I$, let $\theta_i \in H^1(Y)$ such that $\Delta_y \theta_i = \psi_i$. In view of the hypoellipticity of the Laplace operator Δ_y, the function θ_i is of class $\mathcal{C}^\infty$, thus, it belongs to $\mathcal{C}^\infty_{per}(Y)$. Let

$$\theta = \sum_{i\in I} \varphi_i \otimes \theta_i \otimes \chi_i.$$

We have $\theta \in \mathcal{D}(Q) \otimes \mathcal{C}^\infty_{per}(Y) \otimes \mathcal{C}_{per}(Z)$ and $\Delta_y \theta = \psi$. Hence, (2.2) follows and the lemma is proved. □

3. Convergence of the homogenization process

3.1. **Preliminary results.** Let B^ε be the linear operator in $L^2(\Omega)$ with domain

$$D(B^\varepsilon) = \left\{v \in H^1_0(\Omega) : \mathcal{A}^\varepsilon v \in L^2(\Omega)\right\},$$

defined by

$$B^\varepsilon u = \mathbf{i}\mathcal{A}^\varepsilon u \quad \text{ for all } u \in D(B^\varepsilon).$$

In the sequel, we suppose that the coefficients $(a_{ij})_{1\le i,j\le N}$ verify

$$a_{ij} \in W^{1,\infty}\left(\mathbb{R}^N_y\right) \quad (1 \le i,j \le N), \tag{3.1}$$

where $W^{1,\infty}\left(\mathbb{R}^N_y\right)$ is the Sobolev space of functions in $L^\infty\left(\mathbb{R}^N_y\right)$ with their derivatives of order 1. Then B^ε is of dense domain, and skew-adjoint since $\mathcal{A}^\varepsilon$ is self-adjoint (see [4] for more details). Consequently, B^ε is a m-dissipative operator in $L^2(\Omega)$ by virtue of [4, Corollary 2.4.11]. It follows by the Hille-Yosida-Philips theorem that B^ε is the generator of a contraction semi-group $(G^\varepsilon_t)_{t\ge 0}$.

Now, let us check the existence and uniqueness for (1.3)-(1.5). The abstract evolution problem for (1.3)-(1.5) is given by

$$\begin{cases} u'_\varepsilon = B^\varepsilon u_\varepsilon + F_\varepsilon(u_\varepsilon) \text{ in } [0,T] \\ u_\varepsilon(0) = u^0, \end{cases} \tag{3.2}$$

where F_ε is defined in $L^2\left(0,T;L^2(\Omega)\right)$ by

$$F_\varepsilon(v)(t) = \frac{\mathbf{i}}{\varepsilon}\mathcal{V}^\varepsilon(t)\,v(t) - \mathbf{i}f(t) \qquad (t\in[0,T])$$

for all $v\in L^2\left(0,T;L^2(\Omega)\right)$. We have the following proposition.

Proposition 3.1. *Suppose $u^0\in D(B^\varepsilon)$, $f\in\mathcal{C}\left([0,T];L^2(\Omega)\right)$ and*

$$\frac{1}{\varepsilon}\left\|\mathcal{V}^\varepsilon\right\|_\infty \le \beta \text{ for all } \varepsilon>0, \tag{3.3}$$

where β is a positive constant independent of ε. Suppose further that T verifies

$$\beta T<1. \tag{3.4}$$

Then, the abstract evolution problem (3.2) admits a unique solution $u_\varepsilon\in\mathcal{C}([0,T];D(B^\varepsilon))\cap\mathcal{C}^1\left([0,T];L^2(\Omega)\right)$.

Proof. Let us consider the mapping $\mathbf{\Phi}^\varepsilon$ of $L^2\left(0,T;L^2(\Omega)\right)$ into $L^2\left(0,T;L^2(\Omega)\right)$ defined by

$$\mathbf{\Phi}^\varepsilon(v)(t) = G_t^\varepsilon u^0 + \int_0^t G_{t-s}^\varepsilon F_\varepsilon(v)(s)\,ds \qquad (t\in[0,T])$$

for all $v\in L^2\left(0,T;L^2(\Omega)\right)$ (we recall that $(G_t^\varepsilon)_{t>0}$ is the contraction semi-group generated by B^ε). For fixed $\varepsilon>0$, $\mathbf{\Phi}^\varepsilon$ is a contraction in $L^2\left(0,T;L^2(\Omega)\right)$ with Lipschitz constant βT. Indeed, for v and $w\in L^2\left(0,T;L^2(\Omega)\right)$, one has,

$$\left\|\mathbf{\Phi}^\varepsilon(v)(t)-\mathbf{\Phi}^\varepsilon(w)(t)\right\|_{L^2(\Omega)} \le \int_0^t \left\|F_\varepsilon(v)(s)-F_\varepsilon(w)(s)\right\|_{L^2(\Omega)}ds \qquad (t\in[0,T]).$$

Moreover,

$$\left\|F_\varepsilon(v)(s)-F_\varepsilon(w)(s)\right\|_{L^2(\Omega)} \le \beta\left\|v(s)-w(s)\right\|_{L^2(\Omega)} \qquad (s\in[0,T]),$$

by virtue of (3.3). Its follows from the preceding inequalities that

$$\left\|\mathbf{\Phi}^\varepsilon(v)-\mathbf{\Phi}^\varepsilon(w)\right\|_{L^2(0,T;L^2(\Omega))} \le \beta T\left\|v-w\right\|_{L^2(0,T;L^2(\Omega))}$$

with (3.4). Thus, for fixed $\varepsilon>0$, there exists $u_\varepsilon\in L^2\left(0,T;L^2(\Omega)\right)$ such that $\mathbf{\Phi}^\varepsilon(u_\varepsilon)=u_\varepsilon$ and u_ε is unique. On the other hand, we proceed as in [4, Lemma 4.1.1] and we see that, any solution $u_\varepsilon\in\mathcal{C}([0,T];D(B^\varepsilon))\cap\mathcal{C}^1\left([0,T];L^2(\Omega)\right)$ of (3.2) verifies $\mathbf{\Phi}^\varepsilon(u_\varepsilon)=u_\varepsilon$, and conversely. The proposition is proved. □

Let us prove some estimates for (1.3)-(1.5).

Lemma 3.1. *Suppose that the hypotheses of Proposition 3.1 are satisfied, and the coefficients a_{ij} $(1\le i,j\le N)$ are of the form*

$$a_{ij}=a\delta_{ij} \quad (1\le i,j\le N), \tag{3.5}$$

where $a\in W^{1,\infty}\left(\mathbb{R}_y^N\right)$, and δ_{ij} is the Kronecker symbol. Suppose further that

$$f \text{ and } f'\in L^2\left(0,T;L^2(\Omega)\right) \tag{3.6}$$

and

$$\frac{\partial\mathcal{V}}{\partial\tau}\in L^\infty\left(\mathbb{R}_y^N\times\mathbb{R}_\tau\right) \text{ with } \frac{1}{\varepsilon^2}\left\|\left(\frac{\partial\mathcal{V}}{\partial\tau}\right)^\varepsilon\right\|_\infty \le c_0, \tag{3.7}$$

c_0 being a constant independent of ε. Then, there exists a constant $c>0$ independent of ε such that the solution u_ε of (1.3)-(1.5) verifies:

$$\|u_\varepsilon\|_{L^2\left(0,T;H_0^1(\Omega)\right)} \leq c \tag{3.8}$$

and

$$\|u'_\varepsilon\|_{L^2(0,T;H^{-1}(\Omega))} \leq c. \tag{3.9}$$

Before the proof of this lemma, let us make some usefull remarks. Let us put

$$a^\varepsilon(u,v) = \sum_{i,j=}^{N} \int_\Omega a_{ij}^\varepsilon \frac{\partial u}{\partial x_j} \overline{\frac{\partial v}{\partial x_i}} dx \quad \text{for all } u,v \in H^1(\Omega).$$

Remark 3.1. *As $u_\varepsilon \in \mathcal{C}([0,T];D(B^\varepsilon)) \cap \mathcal{C}^1\left([0,T];L^2(\Omega)\right)$, the function $t \to a^\varepsilon(u_\varepsilon(t), u_\varepsilon(t))$ belongs to $\mathcal{C}^1([0,T])$ and*

$$\frac{d}{dt} a^\varepsilon(u_\varepsilon(t), u_\varepsilon(t)) = 2\operatorname{Re}(\mathcal{A}^\varepsilon u_\varepsilon(t), u'_\varepsilon(t)) \text{ for all } t \in [0,T].$$

On the other hand, by (3.7) we have

$$\frac{1}{\varepsilon}\frac{d}{dt}(\mathcal{V}^\varepsilon u_\varepsilon(t), u_\varepsilon(t)) = \frac{1}{\varepsilon^2}\left(\left(\frac{\partial \mathcal{V}}{\partial \tau}\right)^\varepsilon u_\varepsilon(t), u_\varepsilon(t)\right) + \frac{2}{\varepsilon}\operatorname{Re}(\mathcal{V}^\varepsilon u_\varepsilon(t), u'_\varepsilon(t)) \quad (t \in [0,T]),$$

where $(,)$ denotes the scalar product in $L^2(\Omega)$. Further, by (3.6) we have

$$\frac{d}{dt}(f(t), u_\varepsilon(t)) = (f'(t), u_\varepsilon(t)) + (f(t), u'_\varepsilon(t)) \quad (t \in [0,T]).$$

Proof of Lemma 3.1. Taking the scalar product in $L^2(\Omega)$ of (1.3) with u_ε yields

$$\mathbf{i}(u'_\varepsilon(t), u_\varepsilon(t)) + a^\varepsilon(u_\varepsilon(t), u_\varepsilon(t)) + \frac{1}{\varepsilon}(\mathcal{V}^\varepsilon u_\varepsilon(t), u_\varepsilon(t)) = (f(t), u_\varepsilon(t)) \qquad (t \in [0,T]).$$

Using (3.5), we see that $t \to a^\varepsilon(u_\varepsilon(t), u_\varepsilon(t))$ is a real values function. Thus, by the preceding equality we have

$$\operatorname{Re}(u'_\varepsilon(t), u_\varepsilon(t)) = -\operatorname{Re}(\mathbf{i}f(t), u_\varepsilon(t)) \qquad (t \in [0,T]),$$

i.e.,

$$\frac{1}{2}\frac{d}{dt}\|u_\varepsilon(t)\|^2_{L^2(\Omega)} = -\operatorname{Re}(\mathbf{i}f(t), u_\varepsilon(t)) \qquad (t \in [0,T]).$$

Integrating the preceding equality in $[0,t]$ with $t \in [0,T]$ leads to

$$\|u_\varepsilon(t)\|^2_{L^2(\Omega)} \leq \left\|u^0\right\|^2_{L^2(\Omega)} + 2\int_0^T \int_\Omega |f|\,|u_\varepsilon|\,dxdt. \tag{3.10}$$

Moreover,

$$2\int_0^T \int_\Omega |f|\,|u_\varepsilon|\,dxdt \leq \int_0^T \int_\Omega \left(2T|f|^2 + \frac{1}{2T}|u_\varepsilon|^2\right) dxdt.$$

Consequently, an integration on $[0,T]$ of (3.10) leads to

$$\frac{1}{2}\|u_\varepsilon\|^2_{L^2(0,T;L^2(\Omega))} \leq T\left\|u^0\right\|^2_{L^2(\Omega)} + 2T^2\|f\|^2_{L^2(0,T;L^2(\Omega))}. \tag{3.11}$$

It follows from the preceding inequality that the sequence $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $L^2\left(0,T;L^2(\Omega)\right)$. Now, let us prove (3.8). Taking the scalar product in $L^2(\Omega)$ of (1.3) with u'_ε, one as

$$\mathbf{i}\|u'_\varepsilon(t)\|^2_{L^2(\Omega)} + (\mathcal{A}^\varepsilon u_\varepsilon(t), u'_\varepsilon(t)) + \frac{1}{\varepsilon}(\mathcal{V}^\varepsilon u_\varepsilon(t), u_\varepsilon(t)) = (f(t), u'_\varepsilon(t)) \qquad (t \in [0,T]).$$

By the preceding equality we have,

$$\operatorname{Re}\left(\mathcal{A}^{\varepsilon}u_{\varepsilon}\left(t\right),u_{\varepsilon}'\left(t\right)\right)+\frac{1}{\varepsilon}\operatorname{Re}\left(\mathcal{V}^{\varepsilon}u_{\varepsilon}\left(t\right),u_{\varepsilon}\left(t\right)\right)=\operatorname{Re}\left(f\left(t\right),u_{\varepsilon}'\left(t\right)\right)\qquad\left(t\in\left[0,T\right]\right).$$

Thus, using Remark 3.1 leads to

$$\begin{array}{c}\frac{1}{2}\frac{d}{dt}a^{\varepsilon}\left(u_{\varepsilon}\left(t\right),u_{\varepsilon}\left(t\right)\right)+\frac{1}{2\varepsilon}\frac{d}{dt}\left(\mathcal{V}^{\varepsilon}u_{\varepsilon}\left(t\right),u_{\varepsilon}\left(t\right)\right)=\\ \frac{1}{2\varepsilon^{2}}\left(\left(\frac{\partial\mathcal{V}}{\partial\tau}\right)^{\varepsilon}u_{\varepsilon}\left(t\right),u_{\varepsilon}\left(t\right)\right)+\operatorname{Re}\frac{d}{dt}\left(f\left(t\right),u_{\varepsilon}\left(t\right)\right)-\operatorname{Re}\left(f'\left(t\right),u_{\varepsilon}\left(t\right)\right).\end{array}\tag{3.12}$$

An integration on $[0,t]$ of (3.12) yields,

$$\begin{array}{c}\frac{1}{2}a^{\varepsilon}\left(u_{\varepsilon}\left(t\right),u_{\varepsilon}\left(t\right)\right)+\frac{1}{2\varepsilon}\left(\mathcal{V}^{\varepsilon}u_{\varepsilon}\left(t\right),u_{\varepsilon}\left(t\right)\right)-\frac{1}{2}a^{\varepsilon}\left(u^{0},u^{0}\right)-\frac{1}{2\varepsilon}\left(\mathcal{V}^{\varepsilon}\left(0\right)u^{0},u^{0}\right)=\\ \frac{1}{2\varepsilon^{2}}\int_{0}^{t}\left(\left(\frac{\partial\mathcal{V}}{\partial\tau}\right)^{\varepsilon}u_{\varepsilon}\left(s\right),u_{\varepsilon}\left(s\right)\right)ds+\operatorname{Re}\left(f\left(t\right),u_{\varepsilon}\left(t\right)\right)-\operatorname{Re}\left(f\left(0\right),u^{0}\right)-\operatorname{Re}\int_{0}^{t}\left(f'\left(s\right),u_{\varepsilon}\left(s\right)\right)ds.\end{array}\tag{3.13}$$

It follows from (1.2) and (3.13) that, by (3.7) we have

$$\begin{array}{c}\alpha\left\Vert u_{\varepsilon}\left(t\right)\right\Vert_{H_{0}^{1}(\Omega)}^{2}\leq\beta\left\Vert u_{\varepsilon}\left(t\right)\right\Vert_{L^{2}(\Omega)}^{2}+c_{1}\left\Vert u^{0}\right\Vert_{L^{2}(\Omega)}^{2}+\beta\left\Vert u^{0}\right\Vert_{L^{2}(\Omega)}^{2}+c_{0}\left\Vert u_{\varepsilon}\right\Vert_{L^{2}(0,T;L^{2}(\Omega))}^{2}\\ +2\left\Vert f\left(t\right)\right\Vert_{L^{2}(\Omega)}\left\Vert u_{\varepsilon}\left(t\right)\right\Vert_{L^{2}(\Omega)}+2\left\Vert f\left(0\right)\right\Vert_{L^{2}(\Omega)}\left\Vert u^{0}\right\Vert_{L^{2}(\Omega)}+2\left\Vert f'\right\Vert_{L^{2}(0,T;L^{2}(\Omega))}\left\Vert u_{\varepsilon}\right\Vert_{L^{2}(0,T;L^{2}(\Omega))},\end{array}$$

where $c_1=\sup_{1\leq i,j\leq N}\left\Vert a_{ij}\right\Vert_{\infty}$. Integrating on $[0,T]$ the preceding inequality and using (3.11), we see that the sequence $\left(u_{\varepsilon}\right)_{\varepsilon>0}$ is bounded in $L^{2}\left(0,T;H_{0}^{1}\left(\Omega\right)\right)$, and (3.8) follows. Now, we can prove (3.9). By (1.3), we have

$$\mathbf{i}\int_{0}^{T}\left(u_{\varepsilon}'\left(t\right),\overline{v}\left(t\right)\right)dt+\int_{0}^{T}a^{\varepsilon}\left(u_{\varepsilon}\left(t\right),v\left(t\right)\right)dt+\frac{1}{\varepsilon}\int_{0}^{T}\left(\mathcal{V}^{\varepsilon}u_{\varepsilon}\left(t\right),v\left(t\right)\right)dt=\int_{0}^{T}\left(f\left(t\right),v\left(t\right)\right)dt$$

for all $v\in L^{2}\left(0,T;H_{0}^{1}\left(\Omega\right)\right)$. Hence,

$$\begin{array}{c}\left|\int_{0}^{T}\left(u_{\varepsilon}'\left(t\right),\overline{v}\left(t\right)\right)dt\right|\leq c_{1}\left\Vert u_{\varepsilon}\right\Vert_{L^{2}\left(0,T;H_{0}^{1}(\Omega)\right)}\left\Vert v\right\Vert_{L^{2}\left(0,T;H_{0}^{1}(\Omega)\right)}+\\ \beta c_{2}\left\Vert u_{\varepsilon}\right\Vert_{L^{2}(0,T;L^{2}(\Omega))}\left\Vert v\right\Vert_{L^{2}\left(0,T;H_{0}^{1}(\Omega)\right)}+c_{2}\left\Vert f\right\Vert_{L^{2}(0,T;L^{2}(\Omega))}\left\Vert v\right\Vert_{L^{2}\left(0,T;H_{0}^{1}(\Omega)\right)},\end{array}$$

where c_2 is the constant in the Poincaré inequality. It follows from the preceding inequality that

$$\left\Vert u_{\varepsilon}'\right\Vert_{L^{2}(0,T;H^{-1}(\Omega))}\leq c_{1}\left\Vert u_{\varepsilon}\right\Vert_{L^{2}\left(0,T;H_{0}^{1}(\Omega)\right)}+\beta c_{2}\left\Vert u_{\varepsilon}\right\Vert_{L^{2}(0,T;L^{2}(\Omega))}+c_{2}\left\Vert f\right\Vert_{L^{2}(0,T;L^{2}(\Omega))}.$$

Then, by (3.11) and (3.8) we conclude that the sequence $\left(u_{\varepsilon}'\right)_{\varepsilon>0}$ is bounded in $L^{2}\left(0,T;H^{-1}\left(\Omega\right)\right)$. The lemma is proved. □

3.2. **A convergence theorem.** Let us first introduce some functions spaces. We consider the space

$$\mathbb{F}_{0}^{1}=\mathcal{Y}\left(0,T\right)\times L^{2}\left(Q;L_{per}^{2}\left(Z;H_{\#}^{1}\left(Y\right)\right)\right)$$

provided with the norm

$$\left\Vert \mathbf{u}\right\Vert_{\mathbb{F}_{0}^{1}}=\left(\left\Vert u_{0}\right\Vert_{\mathcal{Y}(0,T)}^{2}+\left\Vert u_{1}\right\Vert_{L^{2}\left(Q;L_{per}^{2}\left(Z;H_{\#}^{1}(Y)\right)\right)}^{2}\right)^{\frac{1}{2}}\qquad\left(\mathbf{u}=\left(u_{0},u_{1}\right)\in\mathbb{F}_{0}^{1}\right),$$

which makes it Hilbert space. We consider also the space

$$\mathcal{F}_{0}^{\infty}=\mathcal{D}\left(Q\right)\times\left[\mathcal{D}\left(Q\right)\otimes\left[\left(\mathcal{C}_{per}\left(Y\right)/\mathbb{C}\right)\otimes\mathcal{C}_{per}\left(Z\right)\right]\right]$$

which is a dense subspace of $\mathbb{F}_0^1$. For $\mathbf{u}=(u_0,u_1)$ and $\mathbf{v}=(v_0,v_1)\in H_0^1(\Omega)\times L^2\left(\Omega;L^2_{per}\left(Z;H^1_\#(Y)\right)\right)$, we set

$$\mathfrak{a}(\mathbf{u},\mathbf{v})=\sum_{i,j=1}^{N}\int\int\int_{\Omega\times Y\times Z}a_{ij}(y)\left(\frac{\partial u_0}{\partial x_j}+\frac{\partial u_1}{\partial y_j}\right)\left(\overline{\frac{\partial v_0}{\partial x_i}}+\overline{\frac{\partial v_1}{\partial y_i}}\right)dxdyd\tau.$$

This defines a sesquilinear hermitian form on $\left[H_0^1(\Omega)\times L^2\left(\Omega;L^2_{per}\left(Z;H^1_\#(Y)\right)\right)\right]^2$ which is continuous and verifies

(3.14)
$$\operatorname{Re}\mathfrak{a}(\mathbf{v},\mathbf{v})\geq\alpha\|\mathbf{v}\|^2_{H_0^1(\Omega)\times L^2\left(\Omega;L^2_{per}\left(Z;H^1_\#(Y)\right)\right)}\qquad\left(\mathbf{v}\in H_0^1(\Omega)\times L^2\left(\Omega;L^2_{per}\left(Z;H^1_\#(Y)\right)\right)\right),$$

according to (1.1)-(1.2). Further, we have the following lemma.

Lemma 3.2. *Suppose that the coefficients a_{ij} $(1\leq i,j\leq N)$ verify (3.5), and let $f\in L^2\left(0,T;L^2(\Omega)\right)$ and $\mathcal{V}\in L^\infty\left(\mathbb{R}^N_y\times\mathbb{R}_\tau;\mathbb{R}\right)$. Then the variational problem*

(3.15)
$$\begin{cases}\mathbf{u}=(u_0,u_1)\in\mathbb{F}_0^1 \text{ with } u_0(0)=u^0:\\ \boldsymbol{i}\int_0^T\langle u_0'(t),\overline{v_0}(t)\rangle dt+\int_0^T\mathfrak{a}(\mathbf{u}(t),\mathbf{v}(t))dt+\int\int\int_{Q\times Y\times Z}(u_1\overline{v_0}+u_0\overline{v_1})\mathcal{V}dxdtdyd\tau\\ \qquad=\int_0^T(f(t),v_0(t))dt\\ \text{for all } \mathbf{v}=(v_0,v_1)\in\mathbb{F}_0^1,\end{cases}$$

admits at most one solution ($\langle,\rangle$ is the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$).

Proof. Suppose $\mathbf{u}=(u_0,u_1)$ and $\mathbf{w}=(w_0,w_1)$ are solutions of (3.15). We set $\mathbf{z}=\mathbf{u}-\mathbf{w}$ ($\mathbf{z}=(z_0,z_1)$ with $z_0=u_0-w_0$ and $z_1=u_1-w_1$). By (3.15), we see that $\mathbf{z}$ verifies

(3.16)
$$\mathbf{i}\int_0^T\langle z_0'(t),\overline{v_0}(t)\rangle dt+\int_0^T\mathfrak{a}(\mathbf{z}(t),\mathbf{v}(t))dt+\int\int\int_{Q\times Y\times Z}(z_1\overline{v_0}+z_0\overline{v_1})\mathcal{V}dxdtdyd\tau=0$$

for all $\mathbf{v}=(v_0,v_1)\in\mathbb{F}_0^1$. Taking in particular $\mathbf{v}=\varphi\otimes\mathbf{v}_*$ with $\varphi\in\mathcal{D}(]0,T[)$ and $\mathbf{v}_*=(v_0,v_1)\in H_0^1(\Omega)\times L^2\left(\Omega;L^2_{per}\left(Z;H^1_\#(Y)\right)\right)$ in (3.16), we obtain

$$\mathbf{i}\langle z_0'(t),\overline{v_0}\rangle+\mathfrak{a}(\mathbf{z}(t),\mathbf{v}_*)+\int\int\int_{\Omega\times Y\times Z}(z_1(t)\overline{v_0}+z_0(t)\overline{v_1})\mathcal{V}dxdyd\tau=0\quad(t\in[0,T])$$

for all $\mathbf{v}_*=(v_0,v_1)\in H_0^1(\Omega)\times L^2\left(\Omega;L^2_{per}\left(Z;H^1_\#(Y)\right)\right)$. Thus, choosing $\mathbf{v}_*=\mathbf{z}(t)$ for $t\in[0,T]$ in the preceding equality yields,

(3.17)
$$\mathbf{i}\langle z_0'(t),\overline{z_0}(t)\rangle+\mathfrak{a}(\mathbf{z}(t),\mathbf{z}(t))+\int\int\int_{\Omega\times Y\times Z}(z_1(t)\overline{z_0}+z_0(t)\overline{z_1})\mathcal{V}dxdyd\tau=0\quad(t\in[0,T]).$$

But, according to (3.5), $t\to\mathfrak{a}(\mathbf{z}(t),\mathbf{z}(t))$ is a real values function. Consequently, by the preceding equality we have

$$\operatorname{Re}\langle z_0'(t),\overline{z_0}(t)\rangle=0\qquad(t\in[0,T]),$$

i.e.,

$$\frac{1}{2}\frac{d}{dt}\|z_0(t)\|^2_{L^2(\Omega)}=0\qquad(t\in[0,T]).$$

Hence $z_0(t)=0$ for all $t\in[0,T]$. Then, by (3.14) and (3.17) we see that $\mathbf{z}(t)=0$ for all $t\in[0,T]$, and the lemma follows. □

In the sequel the coefficients a_{ij} $(1 \leq i, j \leq N)$ are assumed to verify the periodicity hypothesis

$$a_{ij}(y + k) = a_{ij}(y) \quad \text{a.e. in } \mathbb{R}^N \ \ (1 \leq i, j \leq N) \tag{3.18}$$

for all $k \in \mathbb{Z}^N$. Moreover, the potential $\mathcal{V}$ is supposed to satisfy

$$\mathcal{V}(y + k, \tau + l) = \mathcal{V}(y, \tau) \quad \text{a.e. in } \mathbb{R}^N \times \mathbb{R} \tag{3.19}$$

for all $(k, l) \in \mathbb{Z}^N \times \mathbb{Z}$, and

$$\int_Y \mathcal{V}(y, \tau)\, dy = 0 \quad \text{a.e. in } \mathbb{R}. \tag{3.20}$$

Therefore the functions a_{ij} $(1 \leq i, j \leq N)$ and $\mathcal{V}$ are respectively Y-periodic and $Y \times Z$-periodic, where $Y = \left(-\frac{1}{2}, \frac{1}{2}\right)^N$ and $Z = \left(-\frac{1}{2}, \frac{1}{2}\right)$. Further, $\mathcal{V}$ is of zero mean value.

Theorem 3.1. *Suppose the hypotheses of Proposition 3.1 and Lemma 3.1 are satisfied. For fixed $\varepsilon > 0$, let u_ε be the solution of (1.3)-(1.5). Then, as $\varepsilon \to 0$, we have:*

$$u_\varepsilon \to u_0 \ \textit{in } \mathcal{Y}(0, T) \textit{-weak,} \tag{3.21}$$

$$u_\varepsilon \to u_0 \ \textit{in } L^2(Q) \textit{-strong} \tag{3.22}$$

and

$$\frac{\partial u_\varepsilon}{\partial x_j} \to \frac{\partial u_0}{\partial x_j} + \frac{\partial u_1}{\partial y_j} \ \textit{in } L^2(Q) \textit{-weak 2-s} \quad (1 \leq i, j \leq N), \tag{3.23}$$

where $\mathbf{u} = (u_0, u_1) \in \mathbb{F}_0^1$ is the unique solution of (3.15).

Proof. According to Lemma 3.1, the sequence $(u_\varepsilon)_{\varepsilon > 0}$ is bounded in $\mathcal{Y}(0, T)$. Hence, if E is a fundamental sequence, by virtue of Theorem 2.2 there are some subsequence E' extracted from E and some vector function $\mathbf{u} = (u_0, u_1) \in \mathbb{F}_0^1$ such that (3.21)-(3.23) hold when $E' \ni \varepsilon \to 0$.

Thus, thanks to Lemma 3.2, the theorem is certainly proved if we can show that $\mathbf{u}$ verifies (3.15).

Indeed, we begin by verifying that $u_0(0) = u^0$ (it is worth recalling that u_0 may be viewed as a continuous mapping of $[0, T]$ into $L^2(\Omega)$).

Let $v \in H_0^1(\Omega)$, and let $\varphi \in \mathcal{C}^1([0, T])$ with $\varphi(T) = 0$. By integration by parts, we have,

$$\int_0^T \langle u_\varepsilon'(t), v \rangle \varphi(t)\, dt + \int_0^T \langle u_\varepsilon(t), v \rangle \varphi'(t)\, dt = -\left\langle u^0, v \right\rangle \varphi(0),$$

since $u_\varepsilon(0) = u^0$. In view of (3.21)-(3.22), we pass to the limit in the preceding equality as $E' \ni \varepsilon \to 0$. We obtain

$$\int_0^T \langle u_0'(t), v \rangle \varphi(t)\, dt + \int_0^T \langle u_0(t), v \rangle \varphi'(t)\, dt = -\left\langle u^0, v \right\rangle \varphi(0).$$

Since φ and v are arbitrary, we see that $u_0(0) = u^0$.

Finally, let us prove the variational equality of (3.15). Fix any arbitrary two functions

$$\psi_0 \in \mathcal{D}(Q) \ \text{ and } \ \psi_1 \in \mathcal{D}(Q) \otimes [(\mathcal{C}_{per}(Y)/\mathbb{C}) \otimes \mathcal{C}_{per}(Z)],$$

and let

$$\psi_\varepsilon = \psi_0 + \varepsilon\psi_1^\varepsilon, \text{ i.e., } \psi_\varepsilon(x,t) = \psi_0(x,t) + \varepsilon\psi_1\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon}\right) \text{ for all } (x,t)\in Q,$$

where $\varepsilon > 0$ is arbitrary. By (1.3), one as

(3.24)
$$\mathbf{i}\int_0^T \left\langle u_\varepsilon'(t), \overline{\psi}_\varepsilon(t)\right\rangle dt + \int_0^T a^\varepsilon(u_\varepsilon(t),\psi_\varepsilon(t))\,dt + \frac{1}{\varepsilon}\int_0^T (\mathcal{V}^\varepsilon u_\varepsilon(t),\psi_\varepsilon(t))\,dt = \int_0^T (f(t),\psi_\varepsilon(t))\,dt.$$

The aim is to pass to the limit in (3.24) as $E' \ni \varepsilon \to 0$. First, we have

$$\int_0^T \left\langle u_\varepsilon'(t),\overline{\psi}_\varepsilon(t)\right\rangle dt = -\int_Q u_\varepsilon \frac{\partial\overline{\psi}_\varepsilon}{\partial t}dxdt = -\int_Q u_\varepsilon\left(\frac{\partial\overline{\psi}_0}{\partial t} + \varepsilon\left(\frac{\partial\overline{\psi}_1}{\partial t}\right)^\varepsilon + \left(\frac{\partial\overline{\psi}_1}{\partial\tau}\right)^\varepsilon\right)dxdt.$$

Thus, in view of (3.22) (and using Definition 2.1), we have,

$$\int_0^T \left\langle u_\varepsilon'(t),\overline{\psi}_\varepsilon(t)\right\rangle dt \to -\int_Q u_0\frac{\partial\overline{\psi}_0}{\partial t}dxdt = \int_0^T \left\langle u_0'(t),\overline{\psi}_0(t)\right\rangle dt$$

as $E' \ni \varepsilon \to 0$, since

$$\int_Q\left(\int\int_{Y\times Z}\frac{\partial\overline{\psi}_1}{\partial\tau}dyd\tau\right)u_0dxdt = 0$$

by virtue of the $Y\times Z$-periodicity of ψ_1.

Next, we have

$$\int_0^T a^\varepsilon(u_\varepsilon(t),\psi_\varepsilon(t))\,dt \to \int_0^T \mathfrak{a}(\mathbf{u}(t),\boldsymbol{\phi}(t))\,dt$$

as $E' \ni \varepsilon \to 0$, where $\boldsymbol{\phi} = (\psi_0,\psi_1)$ (proceed as in the proof of the similar result in [7, p.179]). On the other hand,

$$\frac{1}{\varepsilon}\int_0^T (\mathcal{V}^\varepsilon u_\varepsilon(t),\psi_\varepsilon(t))\,dt = \frac{1}{\varepsilon}\int_Q \mathcal{V}^\varepsilon u_\varepsilon\overline{\psi}_0 dxdt + \int_Q \mathcal{V}^\varepsilon u_\varepsilon\overline{\psi}_1^\varepsilon dxdt. \tag{3.25}$$

In view of Lemma 2.1 and the density of $(\mathcal{C}_{per}(Y)/\mathbb{C})\otimes\mathcal{C}_{per}(Z)$ in $L^2_{per}\left(Z;L^2_{per}(Y)/\mathbb{C}\right)$, we pass to the limit in (3.25) by virtue of (3.19)-(3.20). This yields,

$$\frac{1}{\varepsilon}\int_0^T (\mathcal{V}^\varepsilon u_\varepsilon(t),\psi_\varepsilon(t))\,dt \to \int\int\int_{Q\times Y\times Z}\left(u_1\overline{\psi}_0 + u_0\overline{\psi}_1\right)\mathcal{V}dxdtdyd\tau$$

as $E' \ni \varepsilon \to 0$. Hence, passing to the limit in (3.24) as $E' \ni \varepsilon \to 0$ leads to

(3.26)
$$\begin{array}{c}\mathbf{i}\int_0^T \left\langle u_0'(t),\overline{\psi}_0(t)\right\rangle dt + \int_0^T \mathfrak{a}(\mathbf{u}(t),\boldsymbol{\phi}(t))\,dt + \int\int\int_{Q\times Y\times Z}\left(u_1\overline{\psi}_0 + u_0\overline{\psi}_1\right)\mathcal{V}dxdtdyd\tau \\ = \int_0^T (f(t),\psi_0(t))\,dt\end{array}$$

for all $\boldsymbol{\phi} = (\psi_0,\psi_1)\in\mathcal{F}_0^\infty$. Moreover, since $\mathcal{F}_0^\infty$ is a dense subspace of $\mathbb{F}_0^1$, by (3.26) we see that $\mathbf{u} = (u_0,u_1)$ verifies (3.15). Thanks to the uniqueness of the solution for (3.15) and the fact that the sequence E is arbitrary, we have (3.21)-(3.23) as $\varepsilon \to 0$. The theorem is proved. □

For further needs, we wish to give a simple representation of the function u_1 in Theorem 3.1. For this purpose, let us introduce the form $\widehat{\mathfrak{a}}$ on $L^2_{per}\left(Z;H^1_{\#}(Y)\right)\times L^2_{per}\left(Z;H^1_{\#}(Y)\right)$ defined by

$$\widehat{\mathfrak{a}}(w,v)=\sum_{i,j=}^{N}\int\int_{Y\times Z}a_{ij}\frac{\partial w}{\partial y_j}\overline{\frac{\partial v}{\partial y_i}}dyd\tau$$

for all w, $v\in L^2_{per}\left(Z;H^1_{\#}(Y)\right)$. By virtue of (1.1)-(1.2) and (3.5), the sesquilinear form $\widehat{\mathfrak{a}}$ is continuous, hermitian and coercive with,

$$\widehat{\mathfrak{a}}(v,v)\geq\alpha\left\|v\right\|^2_{L^2_{per}\left(Z;H^1_{\#}(Y)\right)}\text{ for all }v\in L^2_{per}\left(Z;H^1_{\#}(Y)\right).$$

Next, for any indice l with $1\leq l\leq N$, we consider the variational problem

$$\left\{\begin{array}{c}\chi^l\in L^2_{per}\left(Z;H^1_{\#}(Y)\right)\\ \widehat{\mathfrak{a}}\left(\chi^l,v\right)=\sum_{i=1}^N\int\int_{Y\times Z}a_{il}\overline{\frac{\partial v}{\partial y_i}}dyd\tau\\ \text{for all }v\in L^2_{per}\left(Z;H^1_{\#}(Y)\right),\end{array}\right.\tag{3.27}$$

which determines χ^l in a unique manner. Further, let $\eta\in L^2_{per}\left(Z;H^1_{\#}(Y)\right)$ be the unique function defined by

$$\widehat{\mathfrak{a}}(\eta,v)=\int\int_{Y\times Z}\mathcal{V}\overline{v}dyd\tau\quad\text{for all }v\in L^2_{per}\left(Z;H^1_{\#}(Y)\right).\tag{3.28}$$

Lemma 3.3. *Under the hypotheses of Theorem 3.1, we have*

$$u_1(x,t,y,\tau)=-\sum_{j=1}^N\frac{\partial u_0}{\partial x_j}(x,t)\,\chi^j(y,\tau)+\eta(y,\tau)\,u_0(x,t)\tag{3.29}$$

for almost all $(x,t,y,\tau)\in Q\times Y\times Z$.

Proof. In (3.15) choose the particular test function $\mathbf{v}=(v_0,v_1)\in\mathbb{F}^1_0$ with $v_0=0$ and $v_1=\varphi\otimes v$, where $\varphi\in\mathcal{D}(Q)$ and $v\in L^2_{per}\left(Z;H^1_{\#}(Y)\right)$. This yields
(3.30)

$$\widehat{\mathfrak{a}}(u_1(x,t),v)=-\sum_{i,j=1}^N\frac{\partial u_0}{\partial x_j}(x,t)\int\int_{Y\times Z}a_{ij}\overline{\frac{\partial v}{\partial y_i}}dyd\tau+u_0(x,t)\int\int_{Y\times Z}\mathcal{V}\overline{v}dyd\tau$$

almost everywhere in $(x,t)\in Q$ and for all $v\in L^2_{per}\left(Z;H^1_{\#}(Y)\right)$. But it is clear that $u_1(x,t)$ (for fixed $(x,t)\in Q$) is the sole function in $L^2_{per}\left(Z;H^1_{\#}(Y)\right)$ solving the variational equation (3.30). On the other hand, in view of (3.27)-(3.28) it is an easy matter to check that the right hand side of (3.29) solves the same variational equation. Hence the lemma follows immediatly. □

3.3. **The macroscopic homogenized equation.** Our aim here is to derive the initial boundary value problem for u_0. To begin, for $1\leq i,j\leq N$, let

$$q_{ij}=\int_Y a_{ij}dy-\sum_{1=1}^N\int\int_{Y\times Z}a_{il}\frac{\partial\chi^j}{\partial y_l}dyd\tau,$$

$$b_i = -\int\int_{Y\times Z} \chi^i \mathcal{V} dy d\tau - \sum_{j=1}^{N} \int\int_{Y\times Z} a_{ij} \frac{\partial \eta}{\partial y_j} dy d\tau.$$

By (3.5) one as,

$$q_{ij} = \delta_{ij} \int_Y a dy - \int\int_{Y\times Z} a \frac{\partial \chi^j}{\partial y_i} dy d\tau$$

and

$$b_i = -\int\int_{Y\times Z} \chi^i \mathcal{V} dy d\tau - \int\int_{Y\times Z} a \frac{\partial \eta}{\partial y_i} dy d\tau.$$

Fruther, let

$$\mu = \int\int_{Y\times Z} \eta \mathcal{V} dy d\tau.$$

To the coefficients q_{ij} we attach the differential operator $\mathcal{Q}$ on Q mapping $\mathcal{D}'(Q)$ into $\mathcal{D}'(Q)$ ($\mathcal{D}'(Q)$ being the usual space of complex distributions on Q) as

$$\mathcal{Q}u = -\sum_{i,j=}^{N} q_{ij} \frac{\partial^2 u}{\partial x_j \partial x_i} \text{ for all } u \in \mathcal{D}'(Q).$$

Let

$$b = (b_i)_{i=1,\ldots,N}.$$

We consider the following initial boundary value problem:

$$\mathbf{i}\frac{\partial u_0}{\partial t} + \mathcal{Q}u_0 + b\cdot\nabla u_0 + \mu u_0 = f \text{ in } Q = \Omega\times]0,T[\tag{3.31}$$

$$u_0 = 0 \text{ on } \partial\Omega\times]0,T[\tag{3.32}$$

$$u_0(0) = u^0 \text{ in } \Omega. \tag{3.33}$$

The initial boundary value problem (3.31)-(3.33) is the so-called macroscopic homogenized equation.

Lemma 3.4. *Suppose the hypotheses of Proposition 3.1 and Lemma 3.1 are satisfied. Then, the initial boundary value problem (3.31)-(3.33) admits at most one weak solution u_0 in $\mathcal{Y}(0,T)$.*

Proof. It is an easy exercise to show that if $u_0 \in \mathcal{Y}(0,T)$ verifies (3.31)-(3.33) then $\mathbf{u} = (u_0, u_1)$ [with u_1 given by (3.29)] satisfies (3.15). Hence, the unicity in (3.31)-(3.33) follows by Lemma 3.2. □

Theorem 3.2. *Suppose the hypotheses of Proposition 3.1 and Lemma 3.1 are satisfied. For $\varepsilon > 0$, let $u_\varepsilon \in \mathcal{Y}(0,T)$ be defined by (1.3)-(1.5). Then, as $\varepsilon \to 0$, we have $u_\varepsilon \to u_0$ in $\mathcal{Y}(0,T)$-weak, where u_0 is the unique weak solution of (3.31)-(3.33) in $\mathcal{Y}(0,T)$.*

Proof. As in the proof of Theorem 3.1, from any fundamental sequence E one can extract a subsequence E' such that as $E' \ni \varepsilon \to 0$, we have (3.21)-(3.23), and further (3.26) holds for all $\boldsymbol{\phi} = (\psi_0, \psi_1) \in \mathcal{F}_0^\infty$, where $\mathbf{u} = (u_0, u_1) \in \mathbb{F}_0^1$. Now, substituting (3.29) in (3.26) and then choosing therein the $\boldsymbol{\phi}$'s such that $\psi_1 = 0$, a simple computation yields (3.31) with (3.32)-(3.33), of course. Hence the theorem follows by Lemma 3.4 and using of an obvious argument. □

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